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Symmetry methods and group-theoretic ideas are combined with tools from nonlinear analysis to solve finite deformation problems of nonlinear elastomechanics. A key feature of the work is the ability to perform detailed global bifurcation analyses of differential equations with symmetries.

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Symmetry and Global Bifurcation in Nonlinear Solid Mechanics

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CONTENTS Page

1. Introduction

The basic purpose of this investigation was to combine symmetry methods and group–theoretic strategies with methods of nonlinear analysis to solve problems in nonlinear elasto–mechanics. Several projects have been completed. Since the effective start–up date of Grant No. AFOSR–88–0222 (August 1, 1988) papers [H7] – [H9]* have appeared, [H10] – [H15] have been accepted for publication, [H16] has been submitted for publication, and [H17] has been written up. The major results are summarized below.

*References prefaced with "H" are listed in Section 5. General references are in Section 4.

2. Description of Work

Paper [H7] investigates the foundations of equivariance in nonlinear continuum mechanics within the specific framework of nonlinear elasticity. Of course, techniques for exploiting equivariance in local bifurcation theory are now well known [1]. However, the actual symmetries of the governing equations are rarely required in that setting; only a standard representation of the group need be considered for the (Liapunov – Schmidt) reduced problem, cf. [1]. In contrast, the identification of the symmetries of the *differential equations* is crucial to a global analysis, as proposed in [H4] and [H6]. Experience shows that this identification is far from trivial in realistic problems from nonlinear elastomechanics (where we are typically confronted with boundary value problems for quasi–linear *systems* of differential equations or their discretized form), e.g., cf. [H4] – [H7], [H10] and [H14]. The fundamental ingredients of equivariance are uncovered in [H7], which paves the way for future applications of the methods of [H4] and [H6]. Among other things, the work introduces the novel concept of (global) material symmetry of a body, which generalizes and synthesizes the usual notions of material symmetry and homogeneity employed in rational continuum mechanics.

Paper [H8] was written primarily for workers in computational mechanics to demonstrate (by way of example) the crucial importance of performing a perfect bifurcation analysis. There is a tacit but prevalent misconception in that discipline claiming that a bifurcation analysis is rarely necessary in practice, since imperfections (which are always present in the physical system) perturb bifurcation points to limit points (which are readily computed via arclength methods). However, if his limit–point calculations are to predict the worst–case collapse scenario (smallest load magnitude, temperature change, etc., at which there is a loss of all nearby *stable* equilibria), then the analyst must somehow know (*a–priori*) the imperfections to which the system is the most sensitive. Our point in [H8] is that (in the absence of experiment) this critical or *worst–case imperfection* can be identified in a rational manner only with the benefit of a global bifurcation diagram (including secondary and even tertiary bifurcations). In

particular, we show that two small, very similar loading imperfections (having the same symmetry type) applied to a hexagonal dome structure under gravity load induce two drastically different responses – one is the critical response (as predicted by the global bifurcation diagram), while the other yields a limit load *three times greater* than the critical one. Said differently, even an experienced analyst, with an enough intuition to predict which type of symmetry–breaking imperfection is critical, could be off by 300% in his determination of the largest safe load carried by this structure. (It is noteworthy that several real–world, dome–like structures have collapsed in recent years under loads quite a bit less than the "design" load, e.g., [2])

Paper [H9] concentrates on the exploitation of symmetry to *diagonse* and *compute* bifurcation points efficiently in numerical bifurcation problems. It is a natural complement to [H4], in which group–theoretic techniques for the computation of multiple bifurcation points and global solution branches were first introduced. The main theme in [H9] is the *block diagonalization* of the Jacobian matrix along entire solution branches, the computational advantages of which are obvious. A novel feature of the work is the treatment of non–absolutely irreducible representations. At first glance this appears to be a "dry–water" consideration for steady–state bifurcation problems. Indeed, as pointed out in [1; XIII §10(b)], static, one–parameter bifurcation is not generic in the presence of non–absolutely irreducible representations. However, thermodynamic considerations imply that there is often an underlying potential structure in problems from structural mechanics and nonlinearly elasticity. (This is always the case for the internal forces, but the external forces may or may not be conservative.) Indeed, potential operators are ubiquitous in elastostatics, even though they are not "generic".

In [H9] we also establish an equivariant version of Krasnosel'skii's famous theorem [3], incorporating recent generalizations that allow for nonlinear dependence on the parameter, [4],[5]. We show that the linearizations of such problems in the presence of non–absolutely irreducible representations, yield multi–dimensional kernels that cannot be reduced by group–theoretic arguments. Nonetheless, bifurcation can be established via

standard crossing conditions. We close [H9] with an announcement of a theorem that enables the efficient computation of such bifurcation points.

In [H10] we consider a two-parameter bifurcation problem associated with steadily rotating states of a flexible, current-carrying, nonlinearly elastic wire in an ambient magnetic field. As pointed out by a previous investigator [6], the problem has obvious S^1 symmetry, which complicates a steady-state bifurcation analysis (for the same reasons given above in our discussion about [H9], i.e., S^1 has *real* 2-dimensional non-absolutely irreducible representations). However, one of the main points of the paper, is that the problem possesses more subtle symmetries, corresponding to *proper* rotations through π radians about axes perpendicular to the string. When these are combined with S^1, we obtain $O(2) \subset SO(3)$ as the complete symmetry group. Consequently, the nontrivial (helical) solutions are contained in fixed-point subspaces corresponding to subgroups of $O(2)$, and we analyze global solution branches via the approach established in [H6]. We point out here the drastic difference between our *nonplanar* solutions corresponding to $O(2) \subset SO(3)$ and the planar solutions of standard rotating string problems [7],[8], in which $O(2) \subset O(3)$ (generated by reflection) is the appropriate symmetry. The group-theoretic lessons learned here could accommodate a detailed global analysis of similar rod problems [9], as well as any purely mechanical rod problem with "handedness", cf. § 3.3.

In [H11] we consider steady motions of a loop of string, in which the spatial image is a "frozen" curve in 3-space with the material points moving at constant (tangential) speed. In the absence of external forces, we demonstate that *any space curve* (piecewise smooth and of the appropriate length) will serve as such a configuration, for both inextensible chains and nonlinear elastic strings. For a broad class of the latter, we find that the length is a monotonically increasing function of the speed. For planar motions of inextensible chains, this result was given by Routh [10] in the last century. Aside from our generalizations of this little-known result, we explore the possibility of a "preferred"

shaped based upon energy considerations. In particular, for fixed speed (constant energy) within this class of axial motions, we show that the magnitude of the total angular momentum is a maximum when the shape is planar and circular. The beauty of our analysis lies within its close relationship to the classical isoperimetric inequality (which asserts that of all planar, closed curves of a given length, a circle encloses the maximum area).

Motivated by the results of [H11], we consider the overall stability of the *circular* axial motion (of a loop of string) in paper [H12]. Our approach to this problem is based upon the well known idea of minimizing the total energy (within a more general class of perturbations than in [H11]) with constant (prescribed) angular momentum [12], [13], which yields a standard constrained problem in the calculus of variations. In [H12] we demonstrate that: (i) an inextensible circular loop is stable for all nonzero magnitudes of the prescribed angular momentum; (ii) a nonlinearly elastic circular loop is stable for sufficiently small values of the magnitude of the prescribed angular momentum and may or may not loose stability as that parameter increases, depending crucially upon a simple characterization of the constitutive law (tension vs. stretch diagram). This, in turn, leads naturally to a definition of "stiff" and "soft" elastic strings. We show that soft strings loose stability, whereas stiff strings, like inextensible chains, do not.

Based upon this result, the remainder of [H12] addresses symmetry-breaking bifurcation from the circular loop. In consonance with our stability results, we first demonstrate that stiff strings do not yield bifucating solutions. (In paticular, the "linear string" of classical mechanics is stiff). However, we uncover new solutions for soft strings. Specifically, we show that the generality of our model (within the confines of physically reasonable constitutive laws) permits a rich variety of bifurcation phenomena corresponding to noncircular solutions. The criterion for bifurcation yields a simple graphical interpretation, and we demonstrate that the qualitative bifurcation diagram can be deduced immediately from the graph of the constitutive law.

In [H13] we present an efficient computational implementation of the block–diagonalization scheme mentioned above in our description of [H9]. While the mathematical underpinnings of the method are classical [11], its application to large–scale linear(ized) eigenvalue problems in a computational setting has not been addressed (to the best of our knowledge). In [H13] we demonstrate that a novel combination of group–theoretic techniques and substructuring (domain decomposition) ideas enable the reduced eigenvalue blocks to be assembled directly without constructing the full structural matrices – only the mass and stiffness matrices of an appropriate *repeating substructure* are required. We present several examples to demonstrate the tremendous efficiency of our approach.

Motivated by anti–plane shear problems of nonlinear elasticity and by nonlinear diffusion processes, paper [H14] considers a general class of bifurcation problems governed by second–order, quasi–linear, strongley elliptic pde's on orthotropic rectangular domains in $\mathbb{R}^2$. While the Leray–Schauder degree (and hence, Rabinowitz's global bifurcation theorem [21]) is accessible to such problems, (assuming regularity), the real difficulty here is to identify something like nodal properties to deduce global seperation and unboundedness of bifurcating solution branches. (One gets an inkling of the difficulty by viewing the nodal lines of the modes of vibration for a square membrane, cf [22,p. 302].)

For boundary conditions that maintain the rectangular symmetry, we first extend the domain periodically and work in a space of doubly periodic functions. Of course this trick avoids the problem of regularity at the corners. But it also has the virtue of enormously increasing the symmetry group from $Z_2 \times Z_2$ (or D_2) to $O(2) \times O(2)$, which we subsequently exploit. Following the methods of [H6], we then work in appropriate fixed–point subspaces corresponding to the subgroups which characterize the eigenfuctions of the linearization. The striking feature of functions within these fixed–point subspaces is that their nodal lines are fixed. Consequently, on each local branch we can identify a fixed rectangular region (contained in the original domain), upon which the solution is strictly positive (like the eigenfuction). Finally, a subtle application of the maximum

principle (with boundary and corner conditons[23]) enables us to prove that this positivity property is perserved globally on each branch, which, in turn, implies global separation and unboundedness of the branches. To the best of our knowledge, this is the first such detailed global bifurcation analysis of a pde.

Paper [H15] considers the problem of nonspherical solutions of an inflated spherical (rubber) baloon. There is experimental evidence for such states [19]. In particular, it has been reported that, as the total mass of the gas in the baloon is increased, the spherical shape becomes egg–shaped (axisymmetric), but then eventually reverts back to a sphere again as the total mass is increased. Aside from numerical simulations (e.g., [20]), we know of no rigorous bifurcation analysis of this problem. Employing the tools of equivariant bifurcation and singularity theory, we are able to illuminate the conditions of such a bifurcation for a broad class of physically reasonable stored energy functions. In particular, under more specific conditions, we demonstrate the existence of an isola bifurcation (a closed loop of nontrivial solutions bifurcating from the trivial solution) that is consistent with the previously mentioned experimental observations.

We consider a general class of nonlinear eigenvalue problems for *systems* of second–order ode's with O(2) symmetry in [H16]. This work was motivated by problems from the statics of elastic strings and rods and from nonlinear oscillations of discrete mechanical systems. Following [H6], we work in the D_n–fixed–point subspaces of the function space. The crucial property of these subspaces is that they "freeze" the locations of the zero of the solution vector. This in turn enables us to deduce global seperation of branches (similar to the results in [H14]). To the best of my knowledge, this is the first generalization of the results of [15] to a general system of ode's.

Paper [H17] is a natural by–product of our work in [H14]. Specifically, we drop the symmetry assumptions of [H14] and prove a generalization of a well known result for positive elliptic operators, concerning the unboundedness of a bifurcating branch of solutions emanating from a bifurcation point where the eigenfunction of the linearization is positive [35]. The current "state of the art" for such problems typically involves

semilinear boundary value problems with rather stringent positivity assumptions on the mapping, cf. [36]. In contrast, our result is valid for quasi-linear problems with merely a "local" positivity requirement near the trivial solution.

3. Conclusions

The most significant results of this project are those dealing with the connection between symmetry and nodal structure of global solutions of differential equations. This idea enabled us to obtain two generalizations of the celebrated result of Crandall and Rabinowitz [15] for second-order, scalar-valued ode's (where symmetry plays no essential role) to second-order, scalar-valued, elliptic pde's and systems of second-order ode's. To the best of my knowledge, these are the first results of their kind. Moreover, I believe that our results will have numerous applications and generalizations, e.g., to systems of second-order, elliptic pde's with applications to nonlinear elasticity.

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[H20] Computation of Multiple Symmetry Breaking Bifurcation Points of Conservative Systems, in preparation.

[H21] Stability of Large–Amplitude Rotating States of Nonlinearly Elastic Strings (with T. Block), in preparation.

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